Worksheet 18

1. The uniform distrbution $U(a,b)$ has a constant pdf equal to $(b-a)^{-1}$ between $a < b$ and equal to 0 otherwise. Let $U \sim U(0,1)$ and define $Y = U^2$. Find the PDF of Y and determine what distribution (it is one that we have studies already) it comes from. Hint: Remember to write the final equation in terms of y.

2. Let $U \sim U(0,1)$ and define $Y = U^{1/2}$. Find the PDF of Y and determine what distribution (it is one that we have studies already) it comes from.

3. Let $Z \sim N(0,1)$ and consider the random variable $Y \sim Z^2$. We cannot directly apply the change of variables formula because $g(z) = z^2$ is not a one-to-one function (it maps positive numbers to the same number as a negative number). We can fix this by considering a random variable $X = |Z|$ and then defining Y to (equivalently) be equal to X^2. The density of X is just twice the density of a standard normal, but only for positive values of x:

$$f(x) = \frac{\sqrt{2}}{\sqrt{\pi}} e^{-x^2/2}, \quad x > 0.$$

Use the change of variables formula to derive the density of Y, which we will call χ^2_1 as on the handout.

4. The value of $\Gamma(1/2)$ is equal to $\sqrt{\pi}$. Use this fact to manipulate the density you have in the previous question, which we called χ^2_1, is also a form of the Gamma distribution.

5. Let $Z_1, \ldots, Z_n \overset{\text{i.i.d.}}{\sim} N(0,1)$. If we have $Y = \sum_i Z_i^2$, then we say that Y follows a chi-squared distribution with k degrees of freedom. We write this as $Y \sim \chi^2_k$. Using the results from the previous two questions, (a) what is another name for this distribution? (b) What are the mean, variance, and mfg of Y? Hint: The second part should be easy.

6. Let $U \sim U(-\frac{\pi}{2}, \frac{\pi}{2})$ be a random variable. Define $T = tan(U)$. Use the change of variable formula to determine the form of the pdf of T.[1] We have not seen this distribution before. It is called the (standard) Cauchy distribution, and is included on the reference sheet. It is a very interesting distribution because it has no defined mean or variance.

[1] The derivative of $tan^{-1}(t)$ is $1/(1+t^2)$.

7. There is also a two-dimensional change of variables formula. It's not difficult to write-out, but solving it can get messy. It can be used to derive, for example, for independent $U_1 \sim \chi^2_{k_1}$ and $U_2 \sim \chi^2_{k_2}$ the distribution of $F = \frac{U_1/k_1}{U_2/k_2}$. This is called the F-distribution. Or, for an independent $Z \sim N(0, 1)$ and $U \sim \chi_k^2$, the distribution of $T = \frac{Z}{\sqrt{U/k}}$. This is called the Student-T distribution. These are both important distrbutions in statistics, but the derivations are quite messy. What is an adjective describing how happy you are not to have to derive them?