Worksheet 09

1. (Gambler's Ruin) We are going to play a game. Write down the integers 0 through 6 across the length of a piece of paper and place the white checker on the number 1. For reference, write "A" over the number 6 and "B" over the number 0. The game consists of turns in which you roll a six-sided die and move the checker piece one space to the left if the number is a 1 or 2 and one space to the right if the roll is 3 or greater.[1] Player A wins if the checker gets to the 6th space and Player B wins if the checker gets to the 0th space. Use the black checker piece to keep track of the largest number reached during the game; it will be 6 if player A wins and less than 6 otherwise. Before you start, guess what the probability is that Player A will win. Then, play the game 10 times, recording the maximum square reached each time. When you are done add your results on the back whiteboard.

[1] You can play with a 12-sided die by moving to the left with a 4 or fewer and right otherwise.

2. (Gambler's Ruin, cont.) Consider a generalization of the game you just played where the number of spots is $N + 1$. We will write p_i to be the probability that Player A will win if the checker piece is currently on position i. (a) Write down a formula for the p_i in terms of p_{i-i} and p_{i+1}. (b) What are p_0 and p_N? Hint for (a): Condition on the outcome of the first round.

3. (Gambler's Ruin, cont.) Assume that $N = 3$. Write down an equation for p_1 in terms of p_2 and an equation of p_2 in terms of p_1. Combine and solve.

4. (Gambler's Ruin, cont.) Returning to the general N case, a recursive equation of the form $p_i = Ap_{i+1} + Bp_{i-1}$, where $A \neq B$ is called a difference equation. It has solutions of the form (it's a long calculation, but you can prove it using a recursive application of the technique in the previous question):

$$p_i = C + D\left(\frac{B}{A}\right)^i$$

For some constants C and D. Use the values of p_0 and p_N, called the boundary conditions, to find the values of the constants and therefore a general formula for p_i. Does the equation match the simulated probability?

5. (Branching Process) Let's do one more problem of a similar type. Consider an amoeba that every minute either splits into two, dies, or does nothing, each with probability 1/3. Let D be the event that the population of amoeba eventually dies out. Find $\mathbb{P}D$.