Worksheet 04

1. Consider a deck of cards with 5 suits and 15 cards from each suit. How many different hands of 4 cards can you be dealt if we do not care about the order of the cards that we are given?

2. Consider a set of cards with 4 suits/colors and 10 cards in each suit. What is the probability that a set of 5 randomly dealt cards will contain three cards of the same number and another set of two cards that share the same number? In poker this is called a *full house*.

3. Consider a set of cards with 7 suits/colors and 10 cards in each suit. What is the probability that a set of 5 randomly dealt cards will have three cards of the same number but without having either four or five cards of the same type or three cards of one number and two matching cards of another number? That is, you have three cards of equal numbers but the other two are of numbers that are different from each other and from the set of three.

4. Suppose that two evenly matched teams (say team A and team B) make it to the baseball World Series. We can model the outcome of a game as coin flip that has the letter A one side and the letter B on the other side. The series ends as soon as one of the teams has won four games. Thus, it can end as early as the 4th game (a "sweep") or as late as the 7th game, with one team winning its fourth game compared to the other team's three wins. What's the probability that team A wins the in exactly (a) 4 games, (b) 5 games, (c) 6 games, or (d) 7 games? Add the four numbers to see that the sum to 0.5 as you would (hopefully) expect.

5. It is an easy mathematical result that $\binom{n}{k} = \binom{n}{k, n-k}$. Describe *qualitatively* why this equality must.

6. Prove the formula on the handout for the number of partitions of n things into r groups when r is equal to 3. You should see that the general formula for an aribrary r is relatively clear once you see a specific case.

7. Consider a bag with N colored marbles inside of it where K marbles are black and $N - K$ marbles are white. You select n marbles from the bag without replacing them. What the probability that you select exactly k black marbles?[1]

[1] This is the density of something called the hypergeometric distribution. Come back to you solution in a few weeks after we have formalized the concept of a random variable to see the connection.

8. ($\star$) Let's prove Theorem 4.2. I will mostly give you the ideas but want you to understand each of the steps. (a) To start, convince yourself that the number of ways to select k things from a set of n with replacement when we do not care about the order is equivalent to finding the number of non-negative integer solutions to the following equation:

$$k = x_1 + x_2 + \cdots x_n = \sum_i^n x_i, \quad x_i \in \{0, 1, 2, \ldots\}$$

By thinking of x_i as the number of items that were selected from the i'th element. (b) Now, the trickier bit. Consider a sequence of $(n + k - 1)$ boxes. For example consider $n = 5$ and $k = 13$; here are $(n+k-1) = 17$ boxes:

$$\square\square\square\square\square\square\square\square\square\square\square\square\square\square\square\square\square$$

Consider coloring in k of the boxes. Continuing with our example, here are 13 of the boxes filled in:

$$\blacksquare\blacksquare\square\blacksquare\square\blacksquare\blacksquare\blacksquare\square\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare\ \square$$

Now, take all of adjacent black squares, count them, and turn them into their respective counts. If two unshaded boxes appear next to another, put a zero between them. Similarly, if the sequence starts with an unshaded box, put a zero at the front and if it ends with an unshaded box put a zero at the end. For example:

$$2\square 1\square 3\square 7\square 0$$

Why will this always give n numbers that sum to k? (c) Convince yourself that the number of ways of shading in k of the $(n + k - 1)$ boxes is equivalent to the number of non-negative integer solutions to $k = \sum_i^n x_i$. Finally, (d) put all of the parts together to prove Theorem 4.2.