Worksheet 03

1. Assume that you have a cup with m black marbles and m white marbles. Consider selecting a marble, putting it back, and selecting another marble. What is the probability that both marbles are the same color? The answer might be obvious from last class. Think about treating this as a proof that proves your intuition based only on the naïve definition of probability and basic rule of counting, rather than focusing on the numerical answer.

2. Assume that you have a cup with m black marbles and m white marbles for some $m > 1$. Consider selecting one marble and then another marble, without putting the first one back. What is the probability that the two marbles are the same color? Again, treat this as a proof rather than a calculation. What happens when m becomes large?

3. Ignore the issue with leap years and assume that birthdays are evently distributed throughout the year. What is the probability that 23 randomly selected people will have no shared birthdays? Do not try to simplify the result yet; just write it in terms of factorials, powers, and products.[1]

[1] This question is often called the Birthday Problem, the first of many famous probability questions we will study this semester.

4. Take the logarithm of the previous result and simplify as much as possible, writing the result in terms of $lf(\cdot)$, the logarithm of the factorial function. Most programming languages have an quick function to compute the log factorial. For example, $lf(365) = 1792.33$ and $lf(342) = 1657.34$. Using your result, now calculate the decimal version of the result from the previous question. Does the result seem surprising to you?

5. How many ways can you pick a pair of people from a set of 23 total people? ($\star$) How does this help explain the solution to the previous question?

6. Three students get on a bus to downtown at the same time. The bus makes three stops once it arrives in downtown Richmond. If each student randomly decides which stop to disembark, what is the probability that everyone gets off at the same stop?

7. It is a well known result from calculus that the limit of $(1 - 1/n)^n$ as n goes to infinity is e^{-1}. Consider a set of N^2 items for which there are N black items and $N^2 - N$ red items. If we sample N items with replacement from this set, what is the probability that all of them are are red? Assuming that N is sufficently large, write this in terms of e

using the formula given above.

Now, take a deep breath. There are approximately 10^{22} air molecules in a breath of air and approximately 10^{44} air molecules in the atmosphere. Assuming air has had plenty of time to mix around the world in the past two millennia, what is the probability that the breath of air you just took contains a molecule of air that Caesar exhaled in his dying breath?

8. ($\star$) Write a formal proof of Theorem 3.1 using the basic rule of counting and proof by induction. For a proof by induction, you show that something is true for $k = 1$ (trival in this case) and then show that if it is true for k it must be true for $k + 1$.