Worksheet 15 (Solutions)

1. Let $X \sim N(\mu, \sigma^2)$. Show that $\mathbb{E}(X)$ is equal to μ using the moment generating function.

Solution: See solution to the next question.

2. Let $X \sim N(\mu, \sigma^2)$. Show that $Var(X)$ is equal to σ^2 using the moment generating function.

Solution: The solution for both 2 and 3 are given by taking the derivatives of the moment generating function for $X \sim N(\mu, \sigma^2)$:

$$\frac{\partial}{\partial t} m_X(t) = (\mu + \sigma^2 t) \cdot e^{\mu t + \sigma^2 t^2/2}$$
$$\frac{\partial^2}{\partial^2 t} m_X(t) = (\mu + \sigma^2 t)^2 \cdot e^{\mu t + \sigma^2 t^2/2} + (\sigma^2) \cdot e^{\mu t + \sigma^2 t^2/2}$$

Which gives:

$$\begin{aligned} \mathbb{E}X &= \mu \\ \mathbb{E}X^2 &= \mu^2 + \sigma^2 \\ Var(X) &= \mathbb{E}X^2 - [\mathbb{E}X]^2 = \sigma^2. \end{aligned}$$

3. Let $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$ be independent random variables. Let $W = X + Y$. Show that W, as defined above, is a normally distributed random variable. Find its mean and variance. Hint: Use the moment generating function.

Solution: We know that the moment generating function of W is the product of the moment generating functions of X and Y:

$$\begin{aligned} m_W(t) &= m_X(t) \cdot m_Y(t) \\ &= e^{\mu_1 t + \frac{1}{2} \cdot \sigma_1^2 t^2} \cdot e^{\mu_2 t + \frac{1}{2} \cdot \sigma_2^2 t^2} \\ &= e^{(\mu_1 + \mu_2) t + \frac{1}{2} \cdot (\sigma_1^2 + \sigma_2^2) t^2} \end{aligned}$$

This is the mgf of a normal distribution with mean $\mu_1 + \mu_2$ and variance $\sigma_1^2 + \sigma_2^2$. By the uniqueness theorem we have $W \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

4. Let $X = \mu + \sigma Z$ where $Z \sim N(0,1)$. Show that $X \sim N(\mu, \sigma^2)$. Hint: moment generating function!

Solution: We know that:

$$\begin{aligned} m_X(t) &= e^{\mu t} m_Z(\sigma t) \\ &= e^{\mu t} \cdot e^{(\sigma t)^2/2} \\ &= e^{\mu t + t^2\sigma^2/2} \end{aligned}$$

Which completes the result.

5. Let $X \sim N(3,5)$. Write the probability $\mathbb{P}[X > 10]$ as a function of Φ.

Solution: From the previous question, we know that we can write $X = 3 + \sqrt{5}Z$. So, we have:

$$\begin{aligned} \mathbb{P}[X > 10] &= \mathbb{P}[(3 + \sqrt{5}Z) > 10] \\ &= \mathbb{P}[\sqrt{5}Z > 7] \\ &= \mathbb{P}[Z > 7/sqrt5] \\ &= 1 - \mathbb{P}[Z < 7/sqrt5] \\ &= 1 - \Phi[7/sqrt5] \approx 0.9991274. \end{aligned}$$

6. $(\star)$ Let $X \sim N(0,1)$. Show that the moment generating function $m_X(t)$ is equal to $e^{t^2/2}$. The full form on the handout follows from the other results established above.

Solution: By definition:

$$\begin{aligned} m_X(t) &= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2 + xt} dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(z^2 - 2zt + t^2 - t^2)} dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(z-t)^2/2 + t^2/2} dx \\ &= e^{t^2/2} \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(z-t)^2/2} dx \\ &= e^{t^2/2} \end{aligned}$$

The last step comes because the integral is the density of a $N(t,1)$ distributed random variable. The algebraic manipulations in the exponent comes from an application of completing the square.