Worksheet 12 (Solutions)

1. Let $X \sim Bernoulli(p)$. Compute $m_X(t)$.

Solution: This is a rare case where we can compute the mfg directly by just pushing through the definition.

$$\begin{aligned} m_X(t) &= \mathbb{E}e^{tX} \\ &= \mathbb{P}(X=0)\cdot e^{t\cdot 0} + \mathbb{P}(X=1)\cdot e^{t\cdot 1} \\ &= (1-p)\cdot e^{t\cdot 0} + p\cdot e^{t\cdot 1} \\ &= (1-p) + pe^t. \end{aligned}$$

2. Let $X \sim Bernoulli(p)$. Using the value of $m_X(t)$, re-derive the expected value and variance for of X.

Solution: The first derivative the mgf is:

$$\frac{d}{dt}m_X(t) = pe^t.$$

Evaluated at $t = 0$, this gives $\mathbb{E}X = p$, just as we had last time. We also see that the second derivative (and all others, in fact) is the same:

$$\frac{d^2}{dt^2}m_X(t) = pe^t.$$

And therefore, $\mathbb{E}X^2 = p$. Using the variance formula we have

$$Var(X) = \mathbb{E}X^2 - [\mathbb{E}X]^2 = p - p^2 = p(1-p).$$

Again, confirming our results from the last worksheet.

3. Describing $Y \sim Bin(n,p)$ as the sum of n random variables $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} Bin(1,p)$, determine the value of $m_Y(t)$. Hint: This should be easy.

Solution: If $Y = \sum_i X_i$, then Y should have the desired distribution. Then, we see that:

$$\begin{aligned} m_Y(t) &= \sum_{i=1}^n m_{X_i}(t) \\ &= \sum_{i=1}^n \left((1-p) + pe^t\right) \\ &= \left((1-p) + pe^t\right)^n \end{aligned}$$

4. Let $Y \sim Geom(p)$; we want to find $m_Y(t)$.[1] Start by writing down

[1] This is a little harder, but I will break it down into smaller steps for you.

the definition of the mgf as a sum. We want to make the sum look like a geometry series (this is where the name of the distribution comes from). First, factor out a quantity of pe^t. Now, notice that you can write the remaining part as a sum of the form $\sum_{k=0}^{\infty} r^k$. This is a geometric series; when $|r| < 1$ the quantity converges and is equal to $\frac{1}{1-r}$. Use this to determine a closed form of $m_Y(t)$.

Solution: Following the first few steps in the question, we have:

$$\begin{aligned} m_Y(t) &= \sum_{y=1}^{\infty}(1-p)^{y-1} \cdot p \cdot e^{yt} \\ &= pe^t \times \sum_{y=1}^{\infty}(1-p)^{y-1} \cdot e^{(y-1)t} \\ &= pe^t \times \sum_{y=1}^{\infty} \left[(1-p) \cdot e^t\right]^{y-1} \end{aligned}$$

The sum is then a geometric series with $r = (1-p) \cdot e^t$. For $t < 1$ this is guarunteed to converge and we have:[2]

$$m_Y(t) = pe^t \times \left[\frac{1}{1-(1-p)e^t}\right]$$

[2] The mgf does not need to exist for all t for the properties on the handout to hold; they just need to exist for some open interval around zero.

5. Let $Y \sim Geom(p)$. Using the mgf, what is $\mathbb{E}Y$? Hint: Use the chain rule and multiplication rule, not the division rule. This is a bit messy but there are no surprising tricks.

Solution: The derviative is given by:

$$\frac{d}{dt}m_X(t) = pe^t \times \left[1-(1-p)e^t\right]^{-1} - pe^t \times \left[1-(1-p)e^t\right]^{-2} \cdot (1-p)e^t$$

Evaluated at zero, gives:

$$\begin{aligned} \frac{d}{dt}m_X(t)\Big|_{t=0} &= pe^0 \times \left[1-(1-p)e^0\right]^{-1} - pe^0 \times \left[1-(1-p)e^0\right]^{-2} \cdot (1-p)e^0 \\ &= p \cdot p^{-1} - p \cdot p^{-2} \cdot (1-p) \\ &= 1 + \frac{1-p}{p} = 1 + \frac{1}{p} - 1 = \frac{1}{p} \end{aligned}$$

You can take one more derivative to get the second moment and variance as well. It's just more messy algebra. Feel free to use the variance giving on the handout.

6. Let $Y \sim NB(k, p)$. What are $m_Y(t)$ and $\mathbb{E}Y$?

Solution: The negative binomial is just a sum of k independent geometric random variables. The moment generating function is just the geometry mgf raised to the power of k and the expected value is $\frac{k}{p}$.

7. We want to compute the moment generating function for the Poisson distribution. To start, show that if $X \sim Poisson(\lambda)$, then:

$$m_X(t) = e^{-\lambda} \cdot \sum_{k=0}^{\infty} \frac{\lambda^k e^{tk}}{k!}$$

Then, show how to re-write this as:

$$m_X(t) = e^{-\lambda} e^{\lambda e^t} \cdot \sum_{k=0}^{\infty} \frac{(e^t \lambda)^k e^{-(e^t \lambda)}}{k!}$$

Set $\delta = e^t \lambda$ and notice that the value under the sum is a known quantity. Simply the result.

Solution: Given the hints, it should be straightfoward to get the result using the fact that the quantity under the sum is just a Poisson pmf with $\lambda = \delta$ and therefore sums to one. The final answer is just the part on the left of the sum, which we can reorganize as:

$$m_X(t) = e^{\lambda(e^t - 1)}.$$

8. Let $X \sim Poisson(\lambda)$. What is $Var(X)$?

Solution: This derivative is not quite as bad as the geometric one. We have:

$$\frac{d}{dt} m_X(t) = e^{\lambda(e^t - 1)} \cdot \lambda \cdot e^t$$

If you plug in $t = 0$ we see that $\mathbb{E}X = \lambda$, though we also already knew that. For the second derivative we have:

$$\frac{d^2}{d^2 t} m_X(t) = e^{\lambda(e^t - 1)} \cdot \lambda^2 \cdot e^{2t} + e^{\lambda(e^t - 1)} \cdot \lambda \cdot (e^t)$$

And plugging in $t = 0$ gives:

$$\begin{aligned}
\frac{d^2}{d^2 t} m_X(t)\Big|_{t=0} &= e^{\lambda(e^0 - 1)} \cdot \lambda^2 \cdot e^0 + e^{\lambda(e^0 - 1)} \cdot \lambda \cdot (e^0) \\
&= \lambda^2 + \lambda
\end{aligned}$$

And then, the variance is given by:

$$\begin{aligned}
Var(X) &= \mathbb{E}X^2 - [\mathbb{E}X]^2 \\
&= \lambda^2 + \lambda - \lambda^2 = \lambda.
\end{aligned}$$

So, the expected value and variance of the Poisson distribution are the same. Both are equal to λ.

9. ($\star$) Think up a real-life, possibly very contrived, example of where you might see something that follows a Binomial, Bernoulli, Geometric, and Negative Binomial distribution in real life. Try to indicate what the parameters either are or try to approximate them.

Solution: Answers will vary. I will give some examples next time in class.