# Worksheet 10 (Solutions)

**1**. Let $X$ be a random variable with the following probability mass function:

$$p_X(x) = \begin{cases} 0.6, & \text{if } x = 1 \\ 0.2, & \text{if } x = 2 \\ 0.2, & \text{if } x = 3 \end{cases}$$

Find $\mathbb{E}X$.

*Solution:* The expected value is:

$$\begin{aligned} \mathbb{E}X &= 0.6 \cdot 1 + 0.2 \cdot 2 + 0.2 \cdot 3 \\ &= 1.6 \end{aligned}$$

**2**. Consider flipping a four-sided die with sides 1-4. Let $X$ be the outcome of rolling the die once, $Y$ be the independent outcome of rolling the die a second time, and $Z$ be the sum $X + Y$. (a) What are $\mathbb{E}X$, $\mathbb{E}Y$, and $\mathbb{E}Z$? (b) What is the pmf of $X$? (c) Sketch the cdf of $X$.

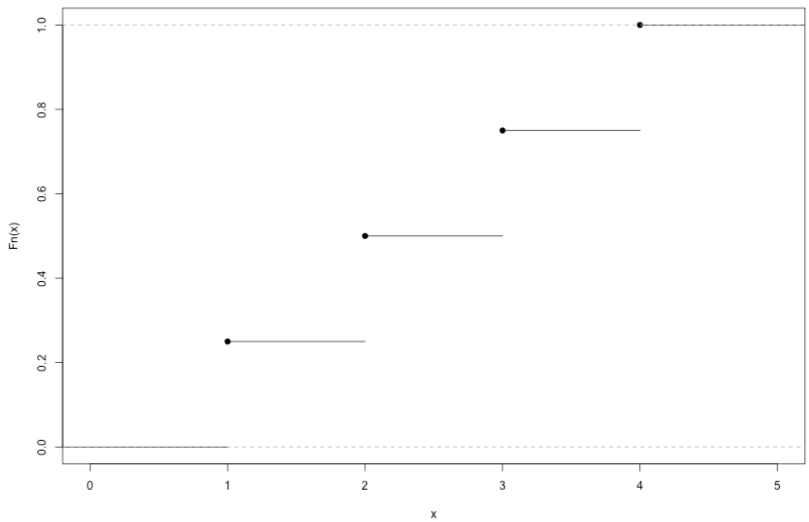
*Solution:* (a) For the expected value, we have:

$$\begin{aligned} \mathbb{E}X &= \frac{1}{4} \cdot 1 + \frac{1}{4} \cdot 2 + \frac{1}{4} \cdot 3 + \frac{1}{4} \cdot 4 \\ &= \frac{1+2+3+4}{4} = \frac{10}{4} = 2.5 \end{aligned}$$

The random variable $Y$ has exactly the same density function and therefore has the same expected value. For $Z$, we have:

$$\mathbb{E}Z = \mathbb{E}\left[X + Y\right] = \mathbb{E}[X] + \mathbb{E}[Y] = 5.$$

(b) The pmf of $X$ is given by $p_X(x) = 1/4$ for all $x \in \{1, 2, 3, 4\}$. Finally, the cdf of $X$ is given by:




**3**. Let $Y$ be a random variable with the following probability mass function:[1]

$$p_Y(y) = \begin{cases} (1-a), & \text{if } y = 0 \\ a, & \text{if } y = 1 \end{cases}$$

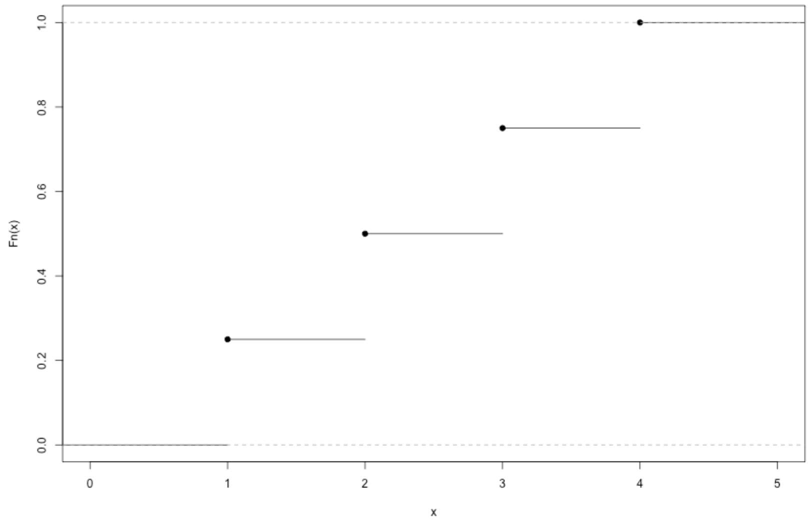
For some $a \in [0, 1]$. Find (a) $\mathbb{E}Y$ and (b) sketch the cdf $F_Y(y)$.

*Solution:* The (a) expected value is simply:

$$\begin{aligned} \mathbb{E}Y &= p(0) \cdot 0 + p(1) \cdot 1 \\ &= a \end{aligned}$$

For (b), I am going to plot this in R using a specific value of $a = 0.6$

[1] I change the variable names here to make this more clear.




In general, the y-axis will have a line at $1 - a$ between the x-values of 0 and 1. It will be 0 for smaller values and 1 for larger values.

**4**. Consider a sequence of $N$ independent tosses of a fair coin. Let $X_1, \ldots, X_n$ be a sequence of independent random variables such that $X_i$ is 1 if the $i$th flip is heads and 0 if it comes up tails. Define $Z = \sum_{i=1}^{n} X_i$, which measures the total number of heads. (a) What is $\mathbb{E}Z$ (Hint: there is an easy way to do this)? (b) What is the pmf of $Z$? (Hint: we have already done this!).

*Solution:* The expected value of each $X_i$ is 0.5, which we can write out as:

$$\begin{aligned}\mathbb{E}X &= \sum_i \mathbb{P}[X_i = x] \cdot x \\ &= \mathbb{P}[X_i = 0] \cdot 0 + \mathbb{P}[X_i = 1] \cdot 1 \\ &= 0 + 0.5 \cdot 1 = 0.5.\end{aligned}$$

The expected value of $Z$ is then:

$$\mathbb{E}Z = \sum_i \mathbb{E}X_i = \sum_i (0.5) = (0.5) \cdot n = \frac{n}{2}.$$

The pmf comes from the combinatorial argument that we have done before. The probability of a specific sequence of 0s and 1s with $z$ 1s is given by $\left(\frac{1}{2}\right)^z \cdot \left(\frac{1}{2}\right)^( n - z)$. There are $\binom{n}{z}$ possible ways to get some sequence with $z$ 1s, and therefore the probability mass function is:

$$\mathbb{P}[Z = z] = \binom{n}{z} \times \left(\frac{1}{2}\right)^z \times \left(\frac{1}{2}\right)^( n - z)$$

**5**. Consider flipping a fair coin until it comes up heads. Let $X$ be a random variable equal to the number of flips that are made. Calculate (a) $p_X(1)$, (b) $p_X(2)$, and (c) $p_X(3)$. (d) Write a general formula for $p_X(n)$. (e) Write down the quantity $\mathbb{E}X$. Notice that the summation is difficult to simplify (you may leave it as is).

*Solution:* For (a-d) have the following formulas:

$$\begin{aligned} p_X(1) &= 1/2 \\ p_X(2) &= (1/2)^2 \\ p_X(3) &= (1/2)^3 \\ &\vdots \\ p_X(n) &= (1/2)^n \end{aligned}$$

The (e) expected value is then given by:

$$\mathbb{E}X = \sum_{i=1}^{\infty} (1/2)^i \cdot i$$

We will see clever tricks for simplifying these in the next worksheet.

**6**. Let $X$ be a random variable defined as follows:

$$p_X(x) = \frac{1}{n}, \quad x \in \{1, 2, \ldots, n\}$$

Calculate $\mathbb{E}X$ and simplify the result.

*Solution:* The expected value is equal to:

$$\begin{aligned} \mathbb{E}X &= \sum_i p(i) \cdot i \\ &= \sum_{i=1}^{n} \frac{1}{n} \cdot i \\ &= \frac{1}{n} \cdot \sum_{i=1}^{n} i \\ &= \frac{1}{n} \cdot \frac{n \cdot (n+1)}{2} \\ &= \frac{n+1}{2} \end{aligned}$$

Using the formula for sum of the first $n$ integers.