Worksheet 09 (Solutions)

1. (Gambler's Ruin) We are going to play a game. Write down the integers 0 through 6 across the length of a piece of paper and place the white checker on the number 1. For reference, write "A" over the number 6 and "B" over the number 0. The game consists of turns in which you roll a six-sided die and move the checker piece one space to the left if the number is a 1 or 2 and one space to the right if the roll is 3 or greater.[1] Player A wins if the checker gets to the 6th space and Player B wins if the checker gets to the 0th space. Use the black checker piece to keep track of the largest number reached during the game; it will be 6 if player A wins and less than 6 otherwise. Before you start, guess what the probability is that Player A will win. Then, play the game 10 times, recording the maximum square reached each time. When you are done add your results on the back whiteboard.

[1] You can play with a 12-sided die by moving to the left with a 4 or fewer and right otherwise.

Solution: Your specific results will, of course, vary.

2. (Gambler's Ruin, cont.) Consider a generalization of the game you just played where the number of spots is $N + 1$. We will write p_i to be the probability that Player A will win if the checker piece is currently on position i. (a) Write down a formula for the p_i in terms of p_{i-i} and p_{i+1}. (b) What are p_0 and p_N? Hint for (a): Condition on the outcome of the first round.

Solution: Let W be the event that A eventually wins if we start on the ith spot and let F be the event that A wins the first round. Then:

$$\begin{aligned} p_i &= \mathbb{P}(W|F)\mathbb{P}F + \mathbb{P}(W|F^c)\mathbb{P}F^c \\ &= \mathbb{P}(W|F) \cdot \frac{2}{3} + \mathbb{P}(W|F^c) \cdot \frac{1}{3} \\ &= p_{i+1} \cdot \frac{2}{3} + p_{i-1} \cdot \frac{1}{3} \end{aligned}$$

Where the last line comes from the fact that if F wins the first round, the probability of winning is equal to the probability of winning if we started on spot $i + 1$, and likewise for lossing and starting on $i - 1$.

We know that $p_0 = 0$ and $p_N = 1$ since the game ends when we get to other spot.

3. (Gambler's Ruin, cont.) Assume that $N = 3$. Write down an equation for p_1 in terms of p_2 and an equation of p_2 in terms of p_1. Combine and solve.

Solution: Plugging into the previous equation, we have:

$$\begin{aligned} p_1 &= p_2 \cdot \frac{2}{3} + p_0 \cdot \frac{1}{3} = p_2 \cdot \frac{2}{3} \\ p_2 &= p_3 \cdot \frac{2}{3} + p_1 \cdot \frac{1}{3} = \frac{2}{3} + p_1 \cdot \frac{1}{3} \end{aligned}$$

Plugging the second into the first gives:

$$\begin{aligned} p_1 &= p_2 \cdot \frac{2}{3} \\ &= \left(\frac{2}{3} + p_1 \cdot \frac{1}{3}\right) \times \frac{2}{3} \\ 9 \cdot p_1 &= 4 + p_1 \cdot 2 \\ 7 \cdot p_1 &= 4 \\ p_1 &= \frac{4}{7} \approx 0.571 \end{aligned}$$

And likewise, $p_2 \approx 0.857$.

4. (Gambler's Ruin, cont.) Returning to the general N case, a recursive equation of the form $p_i = Ap_{i+1} + Bp_{i-1}$, where $A \neq B$ is called a difference equation. It has solutions of the form (it's a long calculation, but you can prove it using a recursive application of the technique in the previous question):

$$p_i = C + D\left(\frac{B}{A}\right)^i$$

For some constants C and D. Use the values of p_0 and p_N, called the boundary conditions, to find the values of the constants and therefore a general formula for p_i. Does the equation match the simulated probability?

Solution: Our solution has this form with $A = \frac{2}{3}$ and $B = \frac{1}{3}$. So,

$$p_i = C + D \cdot 0.5^i$$

We have the two boundary conditions:

$$\begin{aligned} p_0 &= C + D \cdot 0.5^0 = 0 \quad \Rightarrow \quad D = -C \\ p_N &= C + D \cdot 0.5^N = C \cdot (1 - 0.5^N) = 1 \Rightarrow C = \frac{1}{1 - 0.5^N} \end{aligned}$$

Which gives:

$$p_i = \frac{1 - 0.5^i}{1 - 0.5^N}$$

Plugging this into our original question for $N = 6$ and $i = 1$ gives a probability just slightly greater than 0.5. If you are curious, the full set of probabilities p_i are: 0.000, 0.508, 0.762, 0.889, 0.952, 0.984, 1.000.

5. (Branching Process) Let's do one more problem of a similar type. Consider an amoeba that every minute either splits into two, dies, or does nothing, each with probability 1/3. Let D be the event that the population of amoeba eventually dies out. Find $\mathbb{P}D$.

Solution: Consider the first minute. Let E_0 be the event that the amoeba dies, E_1 be the event that the amoeba does nothing, and E_2 be the event that it splits into two. Then:

$$\begin{aligned}\mathbb{P}[D] &= \mathbb{P}[D|E_0] \cdot \mathbb{P}[E_0] + \mathbb{P}[D|E_1] \cdot \mathbb{P}[E_1] + \mathbb{P}[D|E_2] \cdot \mathbb{P}[E_2] \\ &= \frac{1}{3} \times [\mathbb{P}[D|E_0] + \mathbb{P}[D|E_1] + \mathbb{P}[D|E_2]]\end{aligned}$$

If the amoeba dies in the first step, the population must die out, so $\mathbb{P}[D|E_0] = 1$. If nothing happens, then we are right back where we started and $\mathbb{P}[D|E_0] = \mathbb{P}[D]$. If the amoeba splits, then we can consider each of the children as their own independent sub-problem. The chance of each sub-line dying out is $\mathbb{P}[D]$; so the chance of at least one lasting is $\mathbb{P}[D]^2$. Plugging in we get:

$$\mathbb{P}[D] = \frac{1}{3} \times \left[1 + \mathbb{P}[D] + \mathbb{P}[D]^2\right]$$

Setting $x = \mathbb{P}[D]$, we have a quadratic equation:

$$\begin{aligned}3x &= 1 + x + x^2 \\ 0 &= 1 - 2x + x^2 \\ 0 &= (1 - x)^2\end{aligned}$$

We see that the equation has a double root at $x = 1$. So, the probability that the amoeba die out is 1.