Worksheet 05 (Solutions)

1. Consider ordered sequences of the numbers 0 and 1. For example, 0011011 would be a sequence of length 7. How many unique sequences are there of length n?

Solution: This is just sampling with replacement, where $n = 2$ and $k = n$ (we are sampling either a 0 or a 1). So, the number of options is 2^n.

2. We flip a fair coin 10 times. What is the probability that exactly 4 of the flips are heads?

Solution: This is equal to the total number of ways that we can get 4 heads out of 10 flips divided by the total number of options for flipping 10 coins. The total number of possible sequences of heads and tails is 2^{10} for the same logic as the previous question. For the numerator, there are two completely different ways to count the number of ways to have a sequence with exactly 4 heads.

FIRST WAY: Consider the placement of the 4 heads in the sequence of flips. There are ten total spots and we need to pick four, so there are $\binom{10}{4}$ total options.

SECOND WAY: Notice that that a sequence of flips with 4 heads must have 6 tails. Therefore, we are just counting the number of unique permutations of the sequence $HHHHTTTTTT$. If each of the four heads and six tails were unique (they aren't), we would have 10! options. But, for each selected sequence (for example $HTHTHTHTTT$), we can permute the heads 4! ways and the tails 6! ways. So, we are overcounting by a factor of $6! \cdot 4!$ and the total is $10!/(6! \cdot 4!)$. You should notice right away, though that this is the same as$\binom{10}{4}$.[1]

[1] The second way took me a lot more space to explain, but I'm not unconvinced that it's not actually more intuitive.

FINAL SOLUTION: Taking either of the two methods, we then have:

$$\mathbb{P}\left[10 \text{ heads}\right] = \frac{\binom{10}{4}}{2^{10}} = 0.21$$

3. We flip a fair coin 10 times. What is the probability that there are three or fewer heads.

Solution: Continuing from the previous question, we can just count the number of ways that there are exactly 3 heads, exactly 2 heads, exactly 1 heads, and exactly 0 heads and add them. So:

$$\mathbb{P}\left[10 \text{ heads}\right] = \frac{\binom{10}{0} + \binom{10}{1} + \binom{10}{2} + \binom{10}{3}}{2^10} = 0.17$$

Make sure that you don't forget the 10 choose 0 case; that's the one sequence with 0 heads.

4. Based on the solutions to the previous three questions, why must the following identity hold?

$$\sum_{j=0}^{n} \binom{n}{j} = 2^n.$$

Solution: The term $\binom{n}{j}$ is the way of getting exactly j heads in a sequence of n fair coin flips. There are a total of 2^n total coin flips. Summing up all the possible numbers of heads should yield the total number of coin flips, leading to the identity.

5. We have a jar with m white marbles and m black marbles. You start by selecting one marble. In addition to putting it back into the jar, you add another marble of the same color. What is the probability that a second marble selected from the jar will have the same color as the first?

Solution: It doesn't matter which color we start with. In the second round, there will $m+1$ marbles of the color we want to match and m of the other color. So, the probability is:

$$\mathbb{P}\left[\text{match}\right] = \frac{(2m) \cdot (m+1)}{(2m) \cdot (2m+1)} = \frac{m+1}{2m+1}$$

So, as you might have expected, the probability is a little more than 0.5.

6. How many ways are there to permute the letters in the word RAINBOW?

Solution: This is just a straightforward permutation. There are 7 unique letters and therefore $7! = 5040$ ways to do it.

7. How many ways are there to permute the letters in the word STATISTICS?

Solution: This is more difficult because we have three S's, three T's, and two I's. Flipping one S for another does not change the combination.

Much like question 2, there are two ways to do this. The first way is to view this as trying to put each of the letters into ten slots. Going in the order of S, T, I, A, C, we have:

$$\binom{10}{3} \cdot \binom{7}{3} \cdot \binom{4}{2} \cdot \binom{2}{1} \cdot \binom{1}{1} = 50400.$$

The second way is to see that the total number of permutations if all the letters are unique would be 10!. But, we are overcounting by a factor of $3! \cdot 3! \cdot 2!$. So the solution is:

$$\frac{10!}{3! \cdot 3! \cdot 2!} = 50400.$$

And we get the same answer.

There's a third way to get the same answer if you think about a partition of ten things into sets of sizes: 3, 3, 2, 1, 1. It's easy to see that this yields the same numerical answer, but seeing why it's equivalent to a partition is a bit more abstract. I am happy to explain the logic to anyone who would like the understand it.

8. There are m faculty members in the mathematics department and n faculty members in the data science department. We need to form a committee of $k \leq m + n$ faculty members.[2] (a) How many possible committees are there? (b) How many possible committees are there if we know that there are exactly j mathematics faculty on the committee, where j is an integer between 0 and k (inclusive)? (c) Put the two parts together to state and prove and identity for binomial coefficents.

[2] Answer all of the questions in terms of binomial coefficents. Don't try to unpack them.

Solution: For (a), this is just a simple application of the binomial coefficent: $\binom{m+n}{k}$. For (b), we have a multistage experiment in which we pick the j mathematics faculty followed by the $k - j$ data science faculty members. There are, by the basic rule of counting, $\binom{m}{j} \cdot \binom{n}{k-j}$ options. For (c), notice that (b) gives a formula based on the number of mathematics faculty; we can count the total number of committees by summing that formular from $j = 0$ to $j = k$. So:

$$\binom{m+n}{k} = \sum_{j=0}^{k} \binom{m}{j} \cdot \binom{n}{k-j}.$$

This is called Vandermonde's identity. It's very messy to prove directly but easy to see in terms of the story in the question.

9. We counted the probability of getting a set of cards that are all the same color/suit by treating different orderings of the cards as distinct. This is fine if we do it consistently in the numerator and denominator. With our new results, we should see that it is easier to do if we count sampling unordered sets. Take a desk of cards with 7 suits and 20 cards of each suit. What is the probability that a hand of 5 cards all have the same suit?

Solution: We can view this as a two stage experiment where we first select the suit and then select the five cards. The first stage has $\binom{7}{1}$ options and the second stage has $\binom{20}{5}$ options. So:

$$\frac{\binom{7}{1}\cdot\binom{20}{5}}{\binom{20\cdot 7}{5}} = 0.00026$$

You did't need to compute the numeric answer, but I included them above in case you are curious.