Handout 16: Gamma and Beta

Today, we will define two additional, two-parameter families of continuous distributions. While the normal distribution is useful for modeling symettric values over all the reals—and of course, for approximations of independent sums—the two new distributions are useful whenever the bounds of the random variable are in question. Both require a new function, called the **Gamma function**, defined by:

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} \cdot e^{-x} dx$$

The Gamma function has an important recursive relationship that for any positive z, we have: $\Gamma(z+1) = z\Gamma(z)$. This function will be key in defining the normalizing constant in today's distributions.

The **Gamma distribution** is determined by two paramters, usually called α and β. The probability density is given by:[1]

$$f_X(x) = \left[\frac{1}{\Gamma(\alpha)\beta^\alpha}\right] \times \left[x^{\alpha-1}e^{-x/\beta}\right], \quad x > 0.$$

[1] For this and the Beta distribution, I highly encourage looking at the visualizations off the pdf on the course website. Note that the Gamma there uses k and θ in place of α and β, but these have the exact same meaning.

This model has found application in many and diverse studies, including lifetime modeling in reliability studies in engineering and in survival analysis in public health, the modeling of selected physical variables of interest in hydrology and meteorology and the modeling of the waiting time until the occurrence of a fixed number of events. The moment generating function of a random variable $X \sim Gamma(\alpha, \beta)$ is given by:

$$m_X(t) = (1 - \beta t)^{-\alpha}$$

As with the Normal distribution, the moment generating function is key to deriving most results regarding the Gamma. A derivation is give on the second page of this handout.

The second distribution on this handout is the **Beta distribution**, which is also defined in terms of two parameters called α and β. It has a probability density function defined over the interval $[0, 1]$. If $X \sim Beta(\alpha, \beta)$, then we have the pdf:

$$f_X(x) = \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right] \times \left[x^{\alpha-1}(1-x)^{\beta-1}\right], \quad x \in [0, 1].$$

The Beta distribution is particularly important in statistical modeling. It does not have a simple, closed-form moment generating function. We can get the mean and variance directly through integration. Its relationship to other distributions will be shown in the final section when we look at random variable transformations.

A derivation of the moment generating function of a random variable with a Gamma distribution is given by the following:

$$\begin{aligned}
m_X(t) &= \mathbb{E}e^{tX} \\
&= \int_0^\infty e^{tx} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} dx \\
&= \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty x^{\alpha-1} e^{-x(\beta^{-1}-t)} dx \\
&= \frac{1}{\Gamma(\alpha)\beta^\alpha} \Gamma(\alpha) \left(\frac{\beta}{1-\beta t}\right)^\alpha \cdot \int_0^\infty \frac{1}{\Gamma(\alpha)\left(\frac{\beta}{1-\beta t}\right)^\alpha} \cdot x^{\alpha-1} e^{-x/\left(\frac{\beta}{1-\beta t}\right)} dx
\end{aligned}$$

The whole part inside of the integral is the density of a $\Gamma(\alpha, \beta/(1-\beta t))$ variable for $t < 1/\beta$, and therefore integrates to 1. This gives:

$$\begin{aligned}
m_X(t) &= \frac{1}{\Gamma(\alpha)\beta^\alpha} \Gamma(\alpha) \left(\frac{\beta}{1-\beta t}\right)^\alpha \\
&= \left(\frac{1}{1-\beta t}\right)^\alpha \\
&= (1-\beta t)^{-\alpha}
\end{aligned}$$

This completes the result.