Handout 14: Continuous Random Variables

A continuous random variable is a mapping from a probability space into the real line with a continuous cdf.[1] Since there is never a non-zero probability that a continuous random variable takes on any specific value, there is no direct analogue to the pmf. Instead, we can define the probability density function, or pdf, in its place. The pdf $f_X(x)$ is defined such that for any real values $a < b$ we have:

$$\mathbb{P}[a \leq X \leq b] = \int_a^b f_X(x)dx.$$

If we set $a = -\infty$ and $b = x$, we see that the left-hand side becomes the cdf ($F_X(x)$). This gives the following:

$$F_X(x) = \mathbb{P}[X \leq x] = \mathbb{P}[-\infty \leq X \leq x] = \int_{-\infty}^b f_X(x)dx.$$

By the fundamental theorem of calculus, we see that the pdf is just the derivative of the cdf.

The expected value of any function g of X can then be defined by the following integral that mirrors the summation we did with the pmf:

$$\mathbb{E}g(X) = \int_{-\infty}^{\infty} f_X(x) \cdot g(x)\ dx.$$

All of the definitions we have for random variables are either written in terms of the cdf (independence and limits) or the expected value (variance, moments, and the mgf). These apply directly as before to the continuous case, as do all of the results about linearity and independence.

As an example, let $f_X(x) = 2x$ for $x \in (0, 1)$ and be 0 otherwise. Then:

$$\begin{aligned}
F_X(t) &= \int_{-\infty}^t f_X(x)dx = \int_0^t 2xdx = 2 \cdot \left[\frac{1}{2} \cdot x^2\right]_{x=0}^{x=t} = t^2, \quad t \in (0,1) \\
\mathbb{E}X &= \int_{-\infty}^{\infty} f_X(x) \cdot x\, dx = \int_0^1 2x \cdot x\, dx = 2\int_0^1 x^2 dx = 2\left[\frac{1}{3} \cdot x^3\right]_0^1 = \frac{2}{3}. \\
\mathbb{E}X^2 &= \int_{-\infty}^{\infty} f_X(x) \cdot x^2 dx = \int_0^1 2x \cdot x^2 dx = 2\int_0^1 x^3 dx = 2\left[\frac{1}{4} \cdot x^4\right]_0^1 = \frac{2}{4} = \frac{1}{2}. \\
Var(X) &= \mathbb{E}X^2 - (\mathbb{E}X)^2 = \frac{1}{2} - \frac{4}{9} = \frac{1}{18} \\
m_X(t) &= \int_{-\infty}^{\infty} f_X(x) \cdot e^{tx} dx = 2\int_0^1 x \cdot e^{tx} dx
\end{aligned}$$

The moment generating function here does not have a closed-form expression.

[1] These contrast with the discrete distributions, which have a cdf that is are step functions.