Handout 07: Conditional Probability

Two of the most important concepts in the probability are conditional probabilities and independence. Without them, we will find it hard to do too many interesting new things with our new, non-naïve definition of probability. So let's define these concepts today. The conditional probability models the probability of a specific event given the knowledge that another event has occured. We can define it mathematically in the following way.

Definition 7.1 (Conditional Probability) *Let A and B be events from a sample space S such that $\mathbb{P}(B) > 0$. The conditional probability of A given B is written $\mathbb{P}(A|B)$ and defined as:*[1]

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

[1] The naïve definition of probability offers some helpful motivation here. The right-hand side becomes the proportion of outcomes in B for which A also occurs, a more straightforward definition of a conditional probability.

The conditional probability $\mathbb{P}(A|B)$ can be greater than, less then, or exactly equal to the probability $\mathbb{P}$. If the probabilities are the same, this means that knowning about B changes nothing about our knowledge of A. We can use this to motive a definition of independent events. Our actual definition will be slightly more general in order to define independence in the case where one of the sets has probability zero.

Definition 7.2 (Independent Events) *A set of events are called (mutually) independent if the probability of their intersection is equal to the product of their individual probabilities. In particular, two events A and B are independent if $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$.*[2]

[2] When A and B both have non-zero probabilities, this definition is equivalent to $\mathbb{P}(A|B) = \mathbb{P}(A)$, which I find to be much more intuitive.

It is often useful to flip the order of a conditional probability, which requires knowing how to go between $\mathbb{P}(A|B)$ and $\mathbb{P}(B|A)$. We can make use of the famous, and surprisingly easy to prove, Bayes' Theorem.

Theorem 7.1 (Bayes' Theorem) *For any two events A and B such that $\mathbb{P}(A) > 0$ and $\mathbb{P}(B) > 0$, we have:*

$$\mathbb{P}(A|B) = \mathbb{P}(B|A) \times \frac{\mathbb{P}(A)}{\mathbb{P}(B)}.$$

Proof. Rearranging the definition of conditional probability, we see that $\mathbb{P}(A|B) \cdot \mathbb{P}(B)$ is equal to $\mathbb{P}(A \cap B)$. Applying the some logic for $\mathbb{P}(B|A)$, we see that $\mathbb{P}(B|A) \cdot \mathbb{P}(A)$ is also equal to $\mathbb{P}(A \cap B)$. Setting these equal to each other and solving for $\mathbb{P}(A|B)$ yields the result $\blacksquare$.

As we will start seeing today, we can use these definitions to check whether two events are independent. More frequently, though, the importance of the concept of independence is to use it as the assumption behind the construction of new probability spaces.