Handout 02: Naïve Definition of Probability

You likely already have an intuitive definition of how probabilities work. Here, will build off of this type of definition, which we will call naïve probabilities, before developing the formal theory in future notes. Put concisely, the probability of an event can be specified by:

$$\mathbb{P}\left(\text{event}\right) = \frac{\#\{\text{ways the event can occur}\}}{\#\{\text{possible outcomes}\}}$$

For example, what is the probability that a six-sided die rolls an even number? You can surely guess the answer right away, but lets explicitly use our definition:

$$\mathbb{P}\left(\text{even}\right) = \frac{\#\{2, 4, 6\}}{\#\{1, 2, 3, 4, 5, 6\}} = \frac{3}{6} = \frac{1}{2}$$

You will likely notice a shortcoming of this approach; it assumes that all outcomes are equally likely. This deficiency will be addressed in our formal definition. However, in the meantime, there is plenty of intuition that can be obtained sticking to this relatively simple case.

In order to work with our probability definition notice that we need to be able to count collections of things. In the die example this was easy as there were only 6 possible outcomes and we were able to count them by hand. In other examples this will not be possible. To work with these, we need some tools at hand that help us count things.[1]

[1] The field of combinatorics is, more or less, the study of counting. If you find these types of problems fun, reading more about combinatorics is a great next step. We will only scratch the surface here.

Many counting problems can be attacked pretty effectively by thinking of them as multi-stage experiments. In such problems, we are often able to obtain the count we need via the following fundamental rule.

Theorem 2.1 (The Basic Rule of Counting) *If the first stage of an experiment (a generic term we will use for a random, repeatable process) can occur in* n *ways, and the second stage can occur in* k *ways, then the two-stage experiment has* n × k *possible outcomes.*

The basic rule of counting presumes that the number of possibilities at the second stage does not depend on the outcome of the first stage.[2] Another point worth noting here is that the basic rule extends in the obvious way to experiments with any finite number of stages. As long as the number of possible outcomes at each stage does not depend on what happened at earlier stages, the number of outcomes of the experiment as a whole is obtained by multiplication. It also extends to subproblems of an experiment.

[2] The options themselves, though, can depend on the outcome as long as the number is the same. For example, consider drawing two cards from a 52-card deck. The second draw always has 51 options, but the missing card depends on the outcome of the first.

For example, what is the probability that rolling a die twice results in both rolls being even? We could count the combinations directly, but a better approach is to use our new theorem. How many ways can the first roll come up even? 3. How many ways can the second roll come up even? 3. So $n = 3$, $k = 3$, and there are a total of 9 combinations that are even. How many outcomes are there in general? Well, here $n = k = 6$, so there are $6^2 = 36$ combinations. So there is a $9/36 = 0.25$ probability of rolling two straight odd numbers.

We need to develop some useful counting formulas. The formulas we will use most often involve the mathematical symbol "$n!$". The name we use for the symbol $n!$ is "n factorial". Its technical definition is given as:

Definition 2.1 (Factorial) *Let* n *be a positive integer. Then* n*! represents the product of all the integers from* n *on down to one, that is,*

$$n! = n \cdot (n-1) \cdots 2 \cdot 1.$$

In the special case of zero, we have $0! = 1$*.*

Using the basic rule of counting, we can quickly see that the number of ways of arranging n things is equal to $n!$. This is an n stage experiment; in the first stage we pick one item first from all n options, in the second stage we pick from the $n-1$ remaining objects, and so forth. This leads to the definition of $n!$.