Exam 04 (Solutions)

You will have 75-minutes to complete this exam. There are a total of 9 questions, the last one is worth double credit. Questions that include the instruction "Justify that..." require showing your work. You need to give enough steps that show you understand the logic of the answer. You may write your answers on this sheet. Unless otherwise noted, you may start with the moment generating functions given on the reference sheet.

1. [**WS14-01**] Let X be a continuous random variable with a probability mass function equal to $f(x) = C \cdot x^2$ for $x \in [0, 1]$ for some constant C and 0 otherwise. Find the constant C that makes this a valid density function.

Solution: We need the integral of the density to sum to 1, so we get:

$$\begin{aligned} 1 &= \int_{x=0}^{1} Cx^2 dx \\ &= C \cdot \left[\frac{1}{3}x^3\right]_{x=0}^{1} \\ &= C \cdot \left[\frac{1}{3}x^3\right]_{x=0}^{1} \\ &= C \cdot \frac{1}{3} \end{aligned}$$

So $C = 3$.

2. [**WS14-02**] What is the expected value of a random variable with the probability mass function given in question 1?

Solution: The expected value is:

$$\begin{aligned} \mathbb{E}X &= \int_{x=0}^{1} 3x^2 \cdot x dx \\ &= \int_{x=0}^{1} 3x^3 dx \\ &= \left[\frac{3}{4}x^4\right]_{x=0}^{1} \\ &= \frac{3}{4}. \end{aligned}$$

Some of you had a value greater than 1, which should have been obviously wrong since X is never larger than 1.

3. [**WS14-05**] Justify the moment generating function of a random variable X with an exponential distribution with rate λ. Your result should be valid for any $t < \lambda$.

Solution: The cumulative distribution is given by the following integral, where we use the substitution $u = \lambda \cdot x \to du = \lambda dx$:

$$\begin{aligned} F(z) &= \int_0^z f(x)dx \\ &= \int_0^z \lambda \cdot e^{-\lambda \cdot x} dx \\ &= \lambda \cdot \int_0^z e^{-\lambda \cdot x} dx \\ &= \lambda \cdot \frac{1}{\lambda} \cdot \int_0^{\lambda z} e^{-u} du \\ &= \left[-e^{-u}\right]_{x=0}^{\lambda z} \\ &= 1 - e^{-\lambda z} \end{aligned}$$

The desired probability comes from the CDF:[1]

$$\begin{aligned} \mathbb{P}[x \geq 1] &= 1 - \mathbb{P}[x < 1] \\ &= 1 - [1 - e^{-\lambda}] \\ &= e^{-\lambda}. \end{aligned}$$

[1] With continuous random variables, we do not need to be careful about the difference between $\geq$ and $>$ because the probability that the variable will take on an exact value at the endpoint is zero.

4. [**WS14-06**] Justify that the mean of a random variable with an exponential distribution is λ^{-1}. You may start with the form of the moment generating function of the exponential distribution given on the reference sheet.

Solution: These come from the moment generating function. The first two derivatives are:

$$\begin{aligned} \frac{\partial}{\partial t} m_X(t) &= \frac{\partial}{\partial t}\left(\frac{\lambda}{\lambda - t}\right) \\ &= \lambda \cdot (-1) \cdot (\lambda - t)^{-2} \cdot (-1) \\ &= \lambda \cdot (\lambda - t)^{-2} \end{aligned}$$

And,

$$\begin{aligned} \frac{\partial^2}{\partial^2 t} m_X(t) &= \frac{\partial}{\partial t}\left(\lambda \cdot (\lambda - t)^{-2}\right) \\ &= \lambda \cdot (-2) \cdot (\lambda - t)^{-3} \cdot (-1) \\ &= 2\lambda \cdot (\lambda - t)^{-3} \end{aligned}$$

Evaluating at $t = 0$ we get:

$$\begin{aligned} \mathbb{E}X &= \lambda^{-1} \\ \mathbb{E}X^2 &= 2\lambda^{-2} \end{aligned}$$

And, finally, the variance formula yields,

$$\begin{aligned} Var(X) &= 2\lambda^{-2} - \left(\lambda^{-1}\right)^2 \\ &= \lambda^{-2}. \end{aligned}$$

5. [**WS15-01**] Let $X \sim N(\mu, \sigma^2)$. Justify that $\mathbb{E}(X)$ is equal to μ. You may start with the form of the moment generating function of the normal distribution given on the reference sheet.

Solution: The solution is given by taking the derivative of the moment generating function for $X \sim N(\mu, \sigma^2)$:

$$\frac{\partial}{\partial t} m_X(t) = (\mu + \sigma^2 t) \cdot e^{\mu t + \sigma^2 t^2/2}$$

Which gives:

$$\mathbb{E}X = \mu$$

6. [**WS15-03**] Let $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$ be independent random variables. Let $W = X + Y$. Justify that W, as defined above, is a normally distributed random variable. You may start with the form of the moment generating function of the normal distribution given on the reference sheet.

Solution: We know that the moment generating function of W is the product of the moment generating functions of X and Y:

$$\begin{aligned} m_W(t) &= m_X(t) \cdot m_Y(t) \\ &= e^{\mu_1 t + \frac{1}{2} \cdot \sigma_1^2 t^2} \cdot e^{\mu_2 t + \frac{1}{2} \cdot \sigma_2^2 t^2} \\ &= e^{(\mu_1 + \mu_2) t + \frac{1}{2} \cdot (\sigma_1^2 + \sigma_2^2) t^2} \end{aligned}$$

This is the mgf of a normal distribution with mean $\mu_1 + \mu_2$ and variance $\sigma_1^2 + \sigma_2^2$. By the uniqueness theorem we have $W \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

7. [**WS16-01**] Let $X \sim Gamma(\alpha, \beta)$. Justify that $\mathbb{E}X = \alpha\beta$ using the moment generating function. You may start with the form of the moment generating function of the Gamma distribution given on the reference sheet.

Solution: The first derivative of the mgf is given by:

$$\begin{aligned}\frac{d}{dt}\left[m_X(t)\right] &= \frac{d}{dt}\left[(1-\beta t)^{-\alpha}\right] \\ &= -\alpha \cdot (1-\beta t)^{-\alpha-1} \cdot (-\beta) \\ &= \alpha \cdot \beta \cdot (1-\beta t)^{-\alpha-1}\end{aligned}$$

Setting $t = 0$ gives the expected value to be:

$$\begin{aligned}\mathbb{E}X &= \alpha \cdot \beta \cdot (1-\beta(0))^{-\alpha-1} \\ &= \alpha\beta\end{aligned}$$

Which is what we have on our distribution sheet as well.

8. [**WS16-03**] Let $X \sim Gamma(\alpha, \beta)$. Justify that $c \cdot X$ also has a Gamma distribution and find its parameters. You may start with the form of the moment generating function of the Gamma distribution given on the reference sheet.

Solution: Notice that we can manipulate the mgf to move the constant c from the variable X to the argument t:

$$m_{cX}(t) = \mathbb{E}e^{t(cX)} = \mathbb{E}e^{(tc)X} = m_X(ct)$$

So, we then have:

$$\begin{aligned}m_{cX}(t) &= m_X(ct) \\ &= (1-\beta ct)^{-\alpha} \\ &= (1-(\beta c)\cdot t)^{-\alpha}\end{aligned}$$

And we see that this is the mgf of a random variable with a $Gamma(\alpha, \beta c)$ distribution.

9. [**WS16-06**] The normalizing constant in the Beta distribution gives that the following must true for any positive numbers a and b:

$$\left[\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\right] = \int_0^1 \left[x^{a-1}(1-x)^{b-1}\right] dx$$

Let $X \sim Beta(\alpha, \beta)$. Justify that $\mathbb{E}(X) = \frac{\alpha}{\alpha+\beta}$ using the result above.

Solution: Using the integral form of the expected value, we get the following:

$$\begin{aligned}
\mathbb{E}X &= \int_0^1 \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right] \times \left[x^{\alpha-1}(1-x)^{\beta-1}\right] x\,dx \\
&= \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right] \times \int_0^1 \left[x^{\alpha}(1-x)^{\beta-1}\right] dx
\end{aligned}$$

The integrand is the formula from the question with $\alpha \to \alpha + 1$, so we have:

$$\mathbb{E}X = \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right] \times \left[\frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)}\right]$$

And, now we can use the rule that $\Gamma(z+1) = z\Gamma(z)$ to simplify the terms with a $+1$:

$$\begin{aligned}
\mathbb{E}X &= \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right] \times \left[\frac{\alpha\Gamma(\alpha)\Gamma(\beta)}{(\alpha+\beta)\Gamma(\alpha+\beta)}\right] \\
&= \frac{\alpha}{\alpha+\beta}
\end{aligned}$$

And this matches what is on the reference table.