Exam 03 (Solutions)

You will have 75-minutes to complete this exam. You may write your answers on this sheet. There are twenty results spread across six questions. Each result is worth 5 points. **Make sure to label the parts of the question clearly.** You may leave your answers unsimplified.

1. [**WS13-01**] Let X and Y be independent random variables with the following probability mass functions:

$$p_X(x) = \begin{cases} 0.1, & \text{if } x = 1 \\ 0.2, & \text{if } x = 2 \\ 0.7, & \text{if } x = 3 \end{cases} \quad p_Y(y) = \begin{cases} 0.3, & \text{if } y = 1 \\ 0.3, & \text{if } y = 2 \\ 0.4, & \text{if } y = 3 \end{cases}$$

Find (a) $\mathbb{E}X$, (b) $Var(X)$, and (c) the moment generating function of X. Sketch the (d) pmf of X and (e) the cdf of X.

Solution: (a) 2.6, (b) 0.44, (c) $0.1e^t + 0.2e^{2t} + 0.3e^{3t}$, (d) + (e) are plots that come more-or-less directly from the question.[1]

[1] I am not going to work out all of the questions in full on these solutions where they are just the same as Worksheet 13 with different numbers. If you have questions though, please let me know!

2. [**WS13-02**] Define X and Y as in the previous question. What are (a) $\mathbb{E}Y$ and (b) $\mathbb{E}(X + 2 \cdot Y)$?

Solution: (a) 2.1, (b) 6.8

3. [**WS13-03**] Define X and Y as in the previous question. Let $Z = min(X, Y)$. What are the (a) pmf and (b) expected value of Z?

Solution: (a) $\mathbb{P}[Z = 1] = 0.37$, $\mathbb{P}[Z = 2] = 0.35$ $\mathbb{P}[Z = 3] = 0.28$. (b) $\mathbb{E}[Z] = 6.8$.

4. [**WS13-04**] Let $X_1, X_2, \overset{\text{i.i.d.}}{\sim} Bernoulli(p)$ be an infinite sequence of independent random variables. We say that Y follows the *silly negative binomial* distribution if Y counts the number of X_i's that are zero before there are k values of X_i that are one. What are the (a) pmf, (b) expected value, and (c) variance of Y in terms of k and p?

Solution: (a) The pmf is: $\binom{y+k-1}{k-1}(1-p)^y p^k$. Here, there are $y + k$ tosses and we need to decided where the $k - 1$ 1's go, since we already know where the last 1 goes (it is at the end).

(b) The expected value is k less than the normal negative binomial, so: $k/p - k$ or $k(1 - p)/p$.

(c) The variance is unchanged from the normal negative binomial: $k(1 - p)/p^2$.

5. [**WS13-05**] Let $X \sim Bin(n, p)$ and $Y = X/n$. Find the (a) expected value and (b) variance of Y. What are the limits of the (c) expected value and (d) variance as $n \to \infty$?

Solution: The expected value is just:

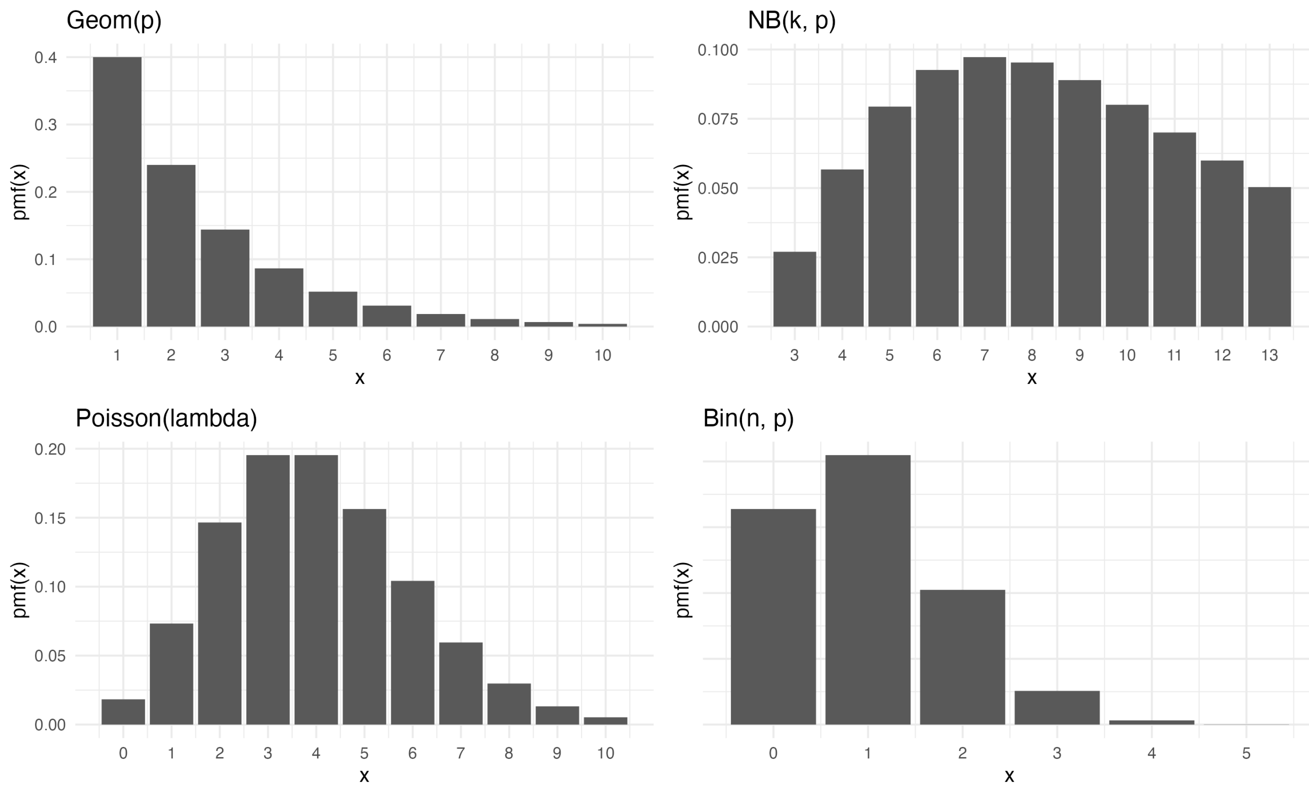
$$\begin{aligned} \mathbb{E}Y &= \mathbb{E}(X/n) \\ &= \frac{1}{n} \cdot \mathbb{E}X \\ &= \frac{np}{n} \\ &= p \end{aligned}$$

And, using the previous result from (1), the variance is just:

$$\begin{aligned} Var(Y) &= Var(X/n) \\ &= \frac{1}{n^2} Var(X) \\ &= \frac{np \cdot (p-1)}{n^2} \\ &= \frac{p \cdot (p-1)}{n} \end{aligned}$$

In the limit, the expected value goes to p and the variance goes to zero. So, with a large sample size, the number of successes limits to p, which makes quite a lot of sense.

6.



[**WS13-06**] Four plots of probability mass functions are given above. The Binomial shows the whole pmf; the others truncate the values for larger x. Estimate as best as possible the unknown parameters for the (a) geometric, (b) Poisson, (c) negative binomial, and (d) binomial distributions.

Solution: For the geometric, we just use the first value. The pmf of 1 is always p; here the pmf is 0.4 so therefore $p = 0.4$.

The smallest possible value for the negative binomial is k, so we know that $k = 3$ in this case. The smallest value of the negative binomial is always p^k. Here the smallest value is about 0.02 and therefore $p \approx (0.02)^{1/3} \approx 0.296$. I actually used a value of 0.4, but anything with this kind of logic is fine.

Exact same logic with the Poisson. The value at zero is always $e^{-\lambda}$. It looks like this is about 0.02 and therefore $\lambda \approx -\log(.02) \approx 3.91$. The actual value is 4; anything remotely close that uses this approach is fine.

We cannot use the same trick from the others because it this one has no scale on the y-axis. The binomial distribution has a positive pmf on the integers 0 through n. Given that the plot goes from 0 to 5, we can deduce that $n = 5$. The binomial also will have its mode (most frequent value) at np (it's mean) rounded to the nearest integer. Since the tallest peak is at 1, this means that p must be between .15 and .25. Anything in that range is fine; a guess of 0.2 would be most natural (and is the one that I used).