Exam 02 (Solutions)

You will have 75-minutes to complete this exam. There are a total of 8 questions; the last two are worth double points. You may write your answers on this sheet. You may leave questions unsimplified.

1. [**WS06-02**] Suppose $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and the sets A, B, and C are given by $A = \{2, 4, 6, 8, 10\}$, $B = \{2, 5, 6, 7, 10\}$, and $C = \{1, 6, 9\}$. Identify each of the following sets:

(a) $A \cup B$
(b) $A \cap B$
(c) $A - B$
(d) $A \cap B \cap C$
(e) $B \cap (A \cup C)^c$

Solution:

(a) $A \cup B$ $\{2, 4, 5, 6, 7, 8, 10\}$
(b) $A \cap B$ $\{2, 6, 10\}$
(c) $A - B$ $\{4, 8\}$
(d) $A \cap B \cap C$ $\{6\}$
(e) $B \cap (A \cup C)^c$ $\{5, 7\}$

2. [**WS06-05**] Based on a (completely made-up) recent survey, it was determined that there is a probability of 0.6 that a randomly selected student's favorite fruit is an apple and a probability of 0.3 that a randomly selected student's favorite dessert is a brownie. It was also determined that there is a probability of 0.2 that randomly selected student reported their favorite fruit as an apple and their favorite dessert to be a brownie. What is the probability that a randomly selected student has a favorite fruit that is not an apple and a favorite dessert that is not a brownie.

Solution: The following table shows all of the values, with the directly given ones in bold.

	$\boldsymbol{A}$	$\boldsymbol{A^c}$	
$\boldsymbol{B}$	**0.2**	0.1	=**0.3**
$\boldsymbol{B^c}$	0.4	0.3	=0.7
	=**0.6**	=0.4	=**1**

Reading off the table, the answer is 0.3.

3. [**WS07-01**] The workers in a particular factory are 65% tall, 70% married, and 45% married and tall. If a worker is selected at random from this factory, find the probability that the worker is a single non-tall person.

Solution: Build a table!

	Tall	Non-Tall	Total
Married	0.45	0.25	0.70
Single	0.20	0.10	0.30
Total	0.65	0.35	1.00

The answer just roll right off the table: 10%.

4. [**WS07-02**] Data science minors are required to take two particular courses: DSST 289 and DSST 389. It is known that the chances of getting an A in DSST 289 is .4 and the chances of getting an A in DSST 389 is .3, while the chances of getting an A in both courses are .1. What are the chances that a randomly selected student will get at least one A in the two courses?

Solution: Another table!

	DSST 289: A	DSST 289: Non-A	Total
DSST 389: A	0.10	0.20	0.30
DSST 389: Non-A	0.30	0.40	0.70
Total	0.40	0.60	1.00

So the answer is $0.10 + 0.20 + 0.30$, or 60%.

5. [**WS07-04**] A student knows the answer to 40 of 60 multiple choice questions on an exam. They select an answer at random (from among 5 possible answers) for the remaining 20 questions. What is the probability that the student actually knew the answer to a particular question that they got correct?

Solution: This is just another conditional probability question that is essentially the same as the previous question. Let K be the student knows the answer and C be the event that it is correct. We know that the probability of $K \cap C^c$ since the student does not get anything wrong that they know. Then, we have another conditional probability in the question:

$$\begin{aligned}\mathbb{P}(C|K^c) &= \frac{\mathbb{P}(C\cap K^c)}{\mathbb{P}(K^c)}\\ 0.2 &= \frac{\mathbb{P}(C\cap K^c)}{1/3}\\ 0.2/3 &= \mathbb{P}(C\cap K^c)\\ 0.0666 &\approx \mathbb{P}(C\cap K^c)\end{aligned}$$

Then, the table is just:

	K	K^c	Total
C	2/3	0.2/3	2.2/3
C^c	0	0.8/3	0.8/3
Total	2.0/3	1.0/3	1

And the question we wanted to answer is:

$$\begin{aligned}\mathbb{P}(K|C) &= \frac{\mathbb{P}(K\cap C)}{\mathbb{P}(C)}\\ &= \frac{2/3}{2.2/3}\\ &= \frac{10}{11}\\ &\approx 0.909.\end{aligned}$$

6. [**WS09-05**] Consider an amoeba that every minute either splits into two, dies, or does nothing, each with probability 1/3. Let D be the event that the population of amoeba eventually dies out. Find $\mathbb{P}D$.

Solution: **[This is exactly the same as the worksheet question]** Consider the first minute. Let E_0 be the event that the amoeba dies, E_1 be the event that the amoeba does nothing, and E_2 be the event that it splits into two. Then:

$$\begin{aligned}\mathbb{P}[D] &= \mathbb{P}[D|E_0]\cdot\mathbb{P}[E_0] + \mathbb{P}[D|E_1]\cdot\mathbb{P}[E_1] + \mathbb{P}[D|E_2]\cdot\mathbb{P}[E_2]\\ &= \frac{1}{3}\times[\mathbb{P}[D|E_0] + \mathbb{P}[D|E_1] + \mathbb{P}[D|E_2]]\end{aligned}$$

If the amoeba dies in the first step, the population must die out, so $\mathbb{P}[D|E_0] = 1$. If nothing happens, then we are right back where we started and $\mathbb{P}[D|E_0] = \mathbb{P}[D]$. If the amoeba splits, then we can consider each of the children as their own independent sub-problem. The chance of each sub-line dying out is $\mathbb{P}[D]$; so the chance of at least one lasting is $\mathbb{P}[D]^2$. Plugging in we get:

$$\mathbb{P}[D] = \frac{1}{3} \times \left[1 + \mathbb{P}[D] + \mathbb{P}[D]^2\right]$$

Setting $x = \mathbb{P}[D]$, we have a quadratic equation:

$$\begin{aligned} 3x &= 1 + x + x^2 \\ 0 &= 1 - 2x + x^2 \\ 0 &= (1 - x)^2 \end{aligned}$$

We see that the equation has a double root at $x = 1$. So, the probability that the amoeba die out is 1.

7. [**WS08-04**] (20 points) Consider a game where there are 8 doors, with a car hidden randomly hidden behind one of the doors. You start by standing in front of one door; then the host opens five of the doors you are not standing in front of, making sure to never reveal the car. In the second round, you can choosen any door to open and see if you have won the car. What is the probability that you will win if you switch from your initial choice to one of the remaining other two doors?

Solution: We can define C_j as the event that the car is behind door j. Let's define W to be the event of winning if we switch. Then:

$$\begin{aligned} \mathbb{P}(W) &= \mathbb{P}(W \cap C_1) + \mathbb{P}(W \cap C_2) + \cdots + \mathbb{P}(S \cap C_8) \\ &= \mathbb{P}(C_1) \cdot \mathbb{P}(W|C_1) + \sum_{i=2}^{8} \mathbb{P}(C_i) \cdot \mathbb{P}(W|C_i) \\ &= \frac{1}{8} \times \mathbb{P}(W|C_1) + \frac{1}{8} \times \sum_{i=2}^{7} \mathbb{P}(W|C_i) \end{aligned}$$

As before, $\mathbb{P}(W|C_1)$ is zero, since not switching will always lose. How about the other doors? If we are conditioning on the event C_i, where $i \neq 1$, we know that switching will result in a win $1/2$ of the time. Why? We are guessing between the remaining open doors that we can switch to; there are two of them, and we are equally likely to get the answer correct. So:

$$\begin{aligned} \mathbb{P}(W) &= \frac{1}{8} \times \sum_{i=2}^{8} \mathbb{P}(W|C_i) \\ &= \frac{1}{8} \times \sum_{i=2}^{8} \frac{1}{2} \\ &= \frac{1}{8} \times \frac{7}{2} = \frac{7}{16} = 0.4375 \end{aligned}$$

You more than triple your chances of winning if you switch to one of the other doors, since the chance of winning without switching is just $1/8 = 0.125$.

8. [**WS08-02**] (20 points) There are two physicians named Dr. A and Dr. Z. Each of them performs two types of procedures: band-aid removal (B) and heart surgery (H). Their recent performance is given by the following tables:

	Heart	Band-Aid
Success	70	10
Failure	20	0

Dr. A

	Heart	Band-Aid
Success	2	81
Failure	8	9

Dr. Z

Define S to be the even that a procedure is successful. Using the empirical probabilities, compute the following:

(a) $\mathbb{P}(S|A)$
(b) $\mathbb{P}(S|Z)$
(c) $\mathbb{P}(S|A \cap H)$
(d) $\mathbb{P}(S|Z \cap H)$
(e) $\mathbb{P}(S|A \cap B)$
(f) $\mathbb{P}(S|Z \cap B)$

Solution:

(a) $\mathbb{P}(S|A) = 80/100$
(b) $\mathbb{P}(S|Z) = 83/100$
(c) $\mathbb{P}(S|A \cap H) = 70/90$
(d) $\mathbb{P}(S|Z \cap H) = 2/10$
(e) $\mathbb{P}(S|A \cap B) = 10/10$
(f) $\mathbb{P}(S|Z \cap B) = 81/90$